Towards Adelic Noncommutative Quantum Mechanics

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Abstract

A motivation of using noncommutative and nonarchimedean geometry on very short distances is given. Besides some mathematical preliminaries, we give a short introduction in adelic quantum mechanics. We also recall to basic ideas and tools embedded in q-deformed and noncommutative quantum mechanics. A rather fundamental approach, called deformation quantization, is noted. A few relations between noncommutativity and nonarchimedean spaces as well as similarities between corresponding quantum theories on them are pointed out. An extended Moyal product in a proposed form of adelic noncommutative quantum mechanics is considered. We suggest some question for future investigations.

1 Introduction

It is widely accepted that standard picture of space-time should be changed around and beyound Planck scale. "Measuring" of spacetime geometry under distances smaller than Planck length l_p is not accesible even to Gedanken experiments. It serves an idea of "quantization" and "discretization" of spacetime and a natural cutoff when using a quantum field theory to describe related phenomena. We are pointing out two possibilties for a reasonable mathematical background of a quantum theory on very small distances, The *first* one comes from the idea of of spacetime coordinates as noncommuting operators

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij}. \tag{1}$$

Some noncommutativity of configuration space should not be a surprise in physics since quantum phase space with the canonical commutation relation

$$[\hat{x}^i, \hat{k}^j] = i\hbar\delta^{ij}, \tag{2}$$

where x^i are coordinates and k^j are the corresponding momenta, is the well-known example of noncommutative (pointless) geometry. This relation is connected in a natural way with the uncertainty principle and a "fuzzy" spacetime pictures at distances $\theta^{1/2}$. Altough, it seems to have good physical sense for $\theta^{1/2} \sim l_p$, characteristic noncommutative distances could be related to gauge couplings [1], closer to observable distances. It should be noted that deriving of uncertainty relation ($\delta x > l_p$) leads to an "strange" notion of quantum line and probably beyound archimedean geometry, because a coordinate always commute with itself [2]!

The *second* promising approach to the physics at the Planck scale, based on non-archimedean geometry was suggested [3]. The simplest way to describe such a geometry (oftenly called also ultrametric) is by using p-adic number fields $\mathbb{Q}_p$ (p is a prime). On the basis of the Ostrovski theorem [4] there is no other nontrivial possibilities (besides field of real numbers $\mathbb{R}$) to complete field of rational numbers $\mathbb{Q}$ in respect to a (nontrivial) norm. Remind that all physical numerical experimental data belong $\mathbb{Q}$.

There have been many interesting applications of p-adic numbers and non-Archimedean geometry in various parts of modern theoretical and mathematical physics (for a review, see [4, 5]). However we restrict ourselves here mainly to p-adic [6] and adelic [7] quantum mechanics (QM). It should be noted that adelic QM have appeared quite useful in quantum cosmology. The appearance of space-time discreteness in adelic formalism (see, e.g. [8]), as well as in noncommutative QM, is an encouragement for the further investigations. We emphasize the role of Feynman's p-adic path integral method on nonarchimedean spaces.

The p-adic analysis and noncommutativity also play a role in some areas of "macroscopic" physics [9]. We list shortly a few of similarities between non-Archimedean and noncommutative structures them and discuss in more details a new observed relation between an ordering on commutative ring in frame of deformation quantization [10] and an ordering on p-adic spaces in an intention to develop path integration on p-adics by "slicening" of trajectories [11].

2 p-Adic numbers and adeles

Any $x \in \mathbb{Q}_p$ can be presented in the form [4]

$$x = p^\nu (x_0 + x_1 p + x_2 p^2 + \cdots) \ , \quad \nu \in \mathbb{Z}, \tag{3}$$

where $x_i = 0, 1, \cdots, p-1$ are digits. p-Adic norm of any term $x_i p^{\nu+i}$ in the canonical expansion (3) is $|x_i p^{\nu+i}|_p = p^{-(\nu+i)}$ and the strong triangle inequality holds, *i.e.* $|a+b|_p \leq \max\{|a|_p, |b|_p\}$. It follows that $|x|_p = p^{-\nu}$ if $x_0 \neq 0$. Derivatives of p-adic valued functions $\varphi : \mathbb{Q}_p \to \mathbb{Q}_p$ are defined as in the real case, but with respect to the p-adic norm. There is no integral $\int \varphi(x) dx$ in a sense of the Lebesgue measure [4], but one can introduce $\int_a^b \varphi(x) dx = \Phi(b) - \Phi(a)$ as a functional of analytic functions $\varphi(x)$, where $\Phi(x)$ is an antiderivative of $\varphi(x)$. In the case of map $f : \mathbb{Q}_p \to \mathbb{C}$ there is well-defined Haar measure. One can use the Gauss integral

$$\int_{\mathbb{Q}_v} \chi_v(ax^2 + bx) dx = \lambda_v(a) |2a|_v^{-\frac{1}{2}} \chi_v\Big(-\frac{b^2}{4a}\Big) \ , \quad a \neq 0, \ v = \infty, 2, 3, 5, \cdots, \tag{4}$$

where index v denotes real ($v = \infty$) and p-adic cases, χ_v is an additive character: $\chi_\infty(x) = \exp(-2\pi i x)$, $\chi_p(x) = \exp(2\pi i \{x\}_p)$, where $\{x\}_p$ is the fractional part of $x \in \mathbb{Q}_p$. $\lambda_v(a)$ is the complex-valued arithmetic function [4]. An adele [12] is an infinite sequence $a = (a_\infty, a_2, ..., a_p, ...)$, where $a_\infty \in \mathbb{R} \equiv \mathbb{Q}_\infty$, $a_p \in \mathbb{Q}_p$ with a restriction that $a_p \in \mathbb{Z}_p$ for all but a finite set S of primes p. The set of all adeles $\mathbb{A}$ may be regarded as a subset of direct topological product $\mathbb{Q}_\infty \times \prod_p \mathbb{Q}_p$. $\mathbb{A}$ is a topological space, and can be considered as a ring with respect to the componentwise addition and multiplication. An elementary function on adelic ring $\mathbb{A}$ is

$$\varphi(x) = \varphi_\infty(x_\infty) \prod_p \varphi_p(x_p) = \prod_v \varphi_v(x_v) \tag{5}$$

with the main restriction that $\varphi(x)$ must satisfy $\varphi_p(x_p) = \Omega(|x_p|_p)$ for all but a finite number of p. Characteristic function on p-adic integers $\mathbb{Z}_p$ is defined by $\Omega(|x|_p) = 1, \ 0 \leq |x|_p \leq 1$ and $\Omega(|x|_p) = 0, \ |x|_p > 1$.

The Fourier transform of the characteristic function (vacuum state) $\Omega(|x_p|)$ is $\Omega(|k_p|)$. All finite linear combinations of elementary functions (5) make the set $\mathcal{D}(\mathbb{A})$ of the Schwartz-Bruhat functions. The Hilbert space $L_2(\mathbb{A})$ is a space of complex-valued functions $\psi_1(x)$, $\psi_2(x), \ldots$, with the scalar product and norm.

3 Adelic quantum mechanics

In foundations of standard QM one usually starts with a representation of the canonical commutation relation (2). In formulation of p-adic QM [6] the multiplication $\hat{q}\psi \to x\psi$ has no meaning for $x \in \mathbb{Q}_p$ and $\psi(x) \in \mathbb{C}$. In the real case momentum and hamiltonian are infinitesimal generators of space and time translations, but, since $\mathbb{Q}_p$ is disconnected field, these infinitesimal transformations become meaningless. However, finite transformations remain meaningful and the corresponding Weyl and evolution operators are p-adically well defined.

Canonical commutation relation (2) in p-adic case can be represented by the Weyl operators ($h = 1$)

$$\hat{Q}_p(\alpha)\psi_p(x) = \chi_p(\alpha x)\psi_p(x) \tag{6}$$

$$\hat{K}_p(\beta)\psi(x) = \psi_p(x + \beta). \tag{7}$$

$$\hat{Q}_p(\alpha)\hat{K}_p(\beta) = \chi_p(\alpha\beta)\hat{K}_p(\beta)\hat{Q}_p(\alpha). \tag{8}$$

It is possible to introduce the family of unitary operators

$$\hat{W}_p(z) = \chi_p(-\frac{1}{2}qk)\hat{K}_p(\beta)\hat{Q}_p(\alpha), \quad z \in \mathbb{Q}_p \times \mathbb{Q}_p, \tag{9}$$

that is a unitary representation of the Heisenberg-Weyl group. Recall that this group consists of the elements (z, α) with the group product $(z, \alpha) \cdot (z', \alpha') = (z + z', \alpha + \alpha' + \frac{1}{2}B(z, z'))$, where $B(z, z') = -kq' + qk'$ is a skew-symmetric bilinear form on the phase space. Dynamics of a p-adic quantum model is described by a unitary operator of evolution $U(t)$ formulated in terms of its kernel $K_t(x, y)$, $U_p(t)\psi(x) = \int_{\mathbb{Q}_p} K_t(x, y)\psi(y)dy$. In this way [6] p-adic QM is given by a triple $(L_2(\mathbb{Q}_p), W_p(z_p), U_p(t_p))$.

Keeping in mind that standard QM can be also given as the corresponding triple, ordinary and p-adic QM can be unified in the form of adelic QM [7]

$$(L_2(A), W(z), U(t)). \tag{10}$$

$L_2(\mathbb{A})$ is the Hilbert space on $\mathbb{A}$, $W(z)$ is a unitary representation of the Heisenberg-Weyl group on $L_2(\mathbb{A})$ and $U(t)$ is a unitary representation of the evolution operator on $L_2(\mathbb{A})$. The evolution operator $U(t)$ is defined by

$$U(t)\psi(x) = \int_{\mathbb{A}} K_t(x, y)\psi(y)dy = \prod_v \int_{\mathbb{Q}_v} K_t^{(v)}(x_v, y_v)\psi^{(v)}(y_v)dy_v. \tag{11}$$

Note that any adelic eigenfunction has the form

$$\Psi(x) = \Psi_\infty(x_\infty) \prod_{p\in S} \Psi_p(x_p) \prod_{p\notin S} \Omega(|x_p|_p), \quad x \in \mathbb{A}, \tag{12}$$

where $\Psi_\infty \in L_2(\mathbb{R})$, $\Psi_p \in L_2(\mathbb{Q}_p)$. In the low-energy limit adelic QM becomes ordinary one.

A suitable way to calculate propagator in p-adic QM is by p-adic generalization of Feynman's path integral [6]. There is no natural ordering on $\mathbb{Q}_p$. However, a bijective continuous map φ from from the set of p-adic numbers $\mathbb{Q}_p$ to the subset $\varphi(\mathbb{Q}_p)$ of real numbers $\mathbb{R}$ [4] was proposed. This map can be defined by (for an older injective version see [11])

$$\varphi(x) = |x|_p \sum_{k=0}^{\infty} x_k p^{-2k}. \tag{13}$$

Than, a linear order on $\mathbb{Q}_p$ is given by the following definition: $x < y$ if $|x|_p < |y|_p$ or when $|x|_p = |y|_p$ there exists such index $m \geq 0$ that digits satisfy $x_0 = y_0, x_1 = y_1, \cdots, x_{m-1} = y_{m-1}$, $x_m < y_m$. One can say: $\varphi(x) > \varphi(y)$ iff $x > y$.

In the case of harmonic oscillator [11], it was shown that there exists the limit

$$K_p(x'',t'';x',t') = \lim_{n\to\infty} K_p^{(n)}(x'',t'';x',t') = \lim_{n\to\infty} N_p^{(n)}(t'',t')$$
$$\times \int_{\mathbb{Q}_p} \cdots \int_{\mathbb{Q}_p} \chi_p\Big(-\frac{1}{h} \sum_{i=1}^{n} \bar{S}(q_i,t_i;q_{i-1},t_{i-1}) \Big) dq_1 \cdots dq_{n-1} \ , \tag{14}$$

where $N_p^{(n)}(t'',t')$ is the corresponding normalization factor for the harmonic oscillator. The subdivision of p-adic time segment $t_0 < t_1 < \cdots < t_{n-1} < t_n$ is made according to linear order on $\mathbb{Q}_p$. In a similar way we have calculated path integrals for a few quantum models. For the references see [13]. Moreover, we were able to obtain general expression for the propagator of the systems with quadratic action (for the details see [14]), without ordering

$$K_p(x'',t'';x',t') = \lambda_p\Big(-\frac{1}{2h} \frac{\partial^2 \bar{S}}{\partial x'' \partial x'} \Big) \Big| \frac{1}{h} \frac{\partial^2 \bar{S}}{\partial x'' \partial x'} \Big|_p^{\frac{1}{2}} \chi_p\Big(-\frac{1}{h} \bar{S}(x'',t'';x',t') \Big). \tag{15}$$

Replacing an index p with v in (15) we can write quantum-mechanical amplitude K in ordinary and all p-adic cases in the same, compact (and adelic) form.

4 Relations between noncommutative and p-adic QM

Noncommutative geometry is geometry which is described by an associative algebra $\mathcal{A}$ whici is usually noncommutative and in which the set of points, if it exists at all, is relegated to a secondary role. Noncommutative spaces have arised in investigation of brane configurations in string and M-theory. Since the one-particle sector of field theories leads to QM, a study of this topic has attracted much of interest. For single particle QM, the corresponding Heisenberg algebra is needed. In addition to (1) and (2) one choose

$$[p^i, p^j] = i\Phi^{ij} \tag{16}$$

There are a lot of possibilities in choosing θ and Φ. Although one can take θ^{ij} and Φ^{ij} are antisymetric nonconstant tensors (matrices), often, the simplest nontrivial case is considered: $\theta^{ij} = const$ and $\Phi^{ij} = 0$. Another realization of noncommutativity is possible by q-deformation of a space, for example, *Manin plane* $xy = qyx$ and q-deformed "classical" phase space $px = qpx$. This approach leads to to latticelike (discrete) structure of space-time [15].

A field $\Psi(x)$ as a function of the noncommuting coordinates x can be used as Schrodinger wave function obeying the free field equation. Other realization, based on star product $(v * \Psi)$ instead of standard multiplication $(V \cdot \Psi)$ of a potential and wave funaction have been considered in corresponding Schrödinger equation too (i.e. see [16]).

The passage from one level of physical theory to more refined another, using what mathematicians call deformation theory is nothing extraordinary new. In a similar way, there is an old idea that QM is some kind of deformed classical mechanics. For a review see [17]. In fact, deformation quantization is closely related to Weyl quantization, shortly sketched in the previous section.

One direction of the investigation led to Moyal bracket and Moyal (star) product, widely used now in noncommutative QM

$$f *_m g = \chi_\infty\left(-\frac{h}{8\pi^2}P\right)(f,g) = fg + \sum_{r=1}^{\infty}\left(\frac{ih}{4\pi}\right)^r P^r(f,g). \tag{17}$$

Several integral formulas have been introduced for the star product and an (formal) parameter of deformation is finally related to some form of Planck constant h. Quantisation can be taken as a deformation of the standard associative and commutative product, now called a star product, of classical

observables driven by the Poissone bracket P. By the intuition, classical mechanics is understood as the limit QM when $h \to 0$.

Some connections between p-adic analysis and quantum deformations has been noticed during the last ten years. It has been observed that the Haar measure on $SU_q(2)$ coincides with the Haar measure on the field of p-adic numbers $\mathbb{Q}_p$ if $q = p^{-1}$ [2]. There is a potential such that the spectrum of the p-adic Schrödinger-like (diffusion) equation [4]

$$D\psi(x) + V(|x|_p)\psi(x) = E\psi(x) \tag{18}$$

is the same one as in the case of q-deformed oscillator found by Biedenharn [18] for$q = 1/p$. For more details see [2].

In a develop of the representation theory for the star product algebras in deformation quantization some non-Archimedean behavior is noted[10]. We find this relation very intrigue and discuss it in more details. Recall, that a star product $*$, on a Poisson manifold (M,π) is a (formal) associative $\mathbb{C}[[\lambda]]$ - bilinear product for $C^\infty(M)[[\lambda]]$

$$f * g = \sum_{r=0}^{\infty} \lambda^r C_r(f,g), \tag{19}$$

with bidiffererential operators C_r. $\mathbb{C}[[\lambda]]$ is an algebra of *formal* (in a formal parameter λ) series $\sum_{r=0}^{\infty} \lambda^r C_r,\ C_r \in \mathbb{C}$ (a convergence of this series is still not under a consideration). $C^\infty(M)[[\lambda]]$ is the space of formal series of smooth functions ($x \in M,\ f_r(x) \in C^\infty(M)$), for a fixed but variable x. With interpretation $\lambda \leftrightarrow \hbar$ we identify $C^\infty(M)[[\lambda]]$ with the algebra of observables of the quantum system corresponding to (M, π). Let R be on an ordered (commutative associative unital) ring R [10], Let us note, that a concept of ordered ring is neccesary one wish to define the positive functionals on a C^* algebra ($C = R(i) = \{a+bi,\ a, b \in R\}$). It is related with Gelfand-Naimark's theorem on commutative spaces. By means of the positive functionals on C^* algebras we can reconstruct the (points on) "starting" manifold, that one on which an algebra of complex function will give the "original" C^* algebra. This is a motivation for generalization on noncommutative space. In this case, the corresponding product of formal functions will be the $*$ product. Now, $R[[\lambda]]$ will be an algebra of formal series on the ordered ring R. Then, if R is an ordered ring, $R[[\lambda]]$ will be ordered in a canonical way, too, by the definition

$$\sum_{r=r_0}^{\infty} \lambda^r a_r > 0, \quad if \quad a_{r_0} > 0. \tag{20}$$

In other words, a formal series in $R[[\lambda]]$ will be a "positive" one, if first nonzero coeficient (an element of ordered ring R is positive). It should be noted that the concept of ordered ring fits naturally to formal power series and thus to Gerstenhaber's deformation theory [19]. Then $R[[\lambda]]$ will be *non-Archimedean*. For example, if we take that $a_0 = -1$, and $a_1 = n$ ($n \in \mathbb{N}$, because for any commutative ordered ring with unit, set of integers is embedded in R, $\mathbb{N} \subset R$, all others coeficients can be zero, we have $-1+n\lambda < 0$ or $n\lambda < 1$ for all $n \in \mathbb{Z}$. The interpretation in formal theory is that the deformation parameter λ is "very small" compared to the other numbers in R. We see that Waldmann's definition of the ordered algebra of formal series on ordered ring R immediately leads to the non-archimedean "structure". It could be a good indication to use ultrametric spaces and p-adics when physical deformation parameter is very small. We would like to underline that Zelanov's ordering by means of map (13) $\mathbb{Q}_p$, i.e. on normed, ultrametric algebra, in frame of p-adic QM for $\lambda = 1/p^2$ is related to the ordering of formal series on ordered rings in the frame of deformation quantization!

5 The adelic Moyal product and noncommutative QM

The presented connections noncommutative vs. "nonarchimedean" QM suggest a need to formulate a quantum theory that may connect as much as possible nonarchimedean and noncommutative effects and structures. At the present level of quantum theory on adeles, a formulation of Noncommutative Adelic QM seems as the most prommising attempt.

An enough simple frame for that might be representation of an algebra of operators (1), (2) and (16). It could be done by linear transformations on corresponding simplectic structure and deformed and extended bilinear product B. Correpondance between classical functions and quantum operators would be provided by Weyl quantisation. An equivalent formulation of noncommutative adelic QM by the triple $(L_2(A_\theta), W_\theta(z), U_\theta(t))$, does not seem to have principles obstacles. In this approach an adele of coordinates x_A would be replaced by a serie of noncommutative operators $\hat{x}_A$, where adelic properties of corresponding eigenvalues is still "conserved".

Now, we has to consider a p-adic and adelic generalization of the Moyal product. Let us consider classical space with coordinates $x^1, x^2, \cdots, x^D$. Let $f(x)$ be a classical function $f(x) = f(x^1, x^2, \cdots, x^D)$. Then, with the respect to the Fourier transformations and the usual Weyl quantization, we

have

$$\hat{f}(x) = \int_{\mathbb{Q}_\infty^D} dk\ \chi_\infty(-k\hat{x})\tilde{f}(k) \equiv f(\hat{x}). \tag{21}$$

Let us now have two classical functions $f(x)$ and $g(x)$ and we are interested in operator product $\hat{f}(x)\hat{g}(x)$. In the real case this operator product is

$$(\hat{f}\cdot\hat{g})(x) = \int\int dkdk'\ \chi_\infty(-k\hat{x})\chi_\infty(-k'\hat{x})\tilde{f}(k)\tilde{g}(k'). \tag{22}$$

Using the Baker-Campbell-Hausdorff formula, the relation (1) and then the coordinate representation one finds the Moyal product in the form

$$(f * g)(x) = \int_{\mathbb{Q}_p^D}\int_{\mathbb{Q}_p^D} dkdk'\ \chi_v\left(-(k+k')x + \frac{1}{2}k_ik'_j\theta^{ij}\right)\tilde{f}(k)\tilde{g}(k'). \tag{23}$$

Note that we already used our generalization from $\mathbb{Q}_\infty$ to $\mathbb{Q}_v$. In the real case we use $k_i \to -(i/2\pi)(\partial/\partial x^i)$ and obtain the well known form $(f*g)(x) = \chi_\infty\left(-\frac{\theta^{ij}}{2(2\pi)^2}\frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j}\right)f(y)g(z)|_{y=z=x}$. In the p-adic case such an straightforward generalization is not possible (but, some kind of psudodifferentiaton could be useful). Thus, as the p-adic Moyal product we take

$$(f * g)(x) = \int_{\mathbb{Q}_p^D}\int_{\mathbb{Q}_p^D} dkdk'\ \chi_p(-(x^ik_i + x^jk'_j) + \frac{1}{2}k_ik'_j\theta^{ij})\tilde{f}(k)\tilde{g}(k'). \tag{24}$$

We can writedown the adelic Moyal product of "classical" adelic functions $f_A = (f_\infty, f_2, ..., f_p, ...)$, $g_A = (g_\infty, g_2, ..., g_p, ...)$ on $\mathbb{R}\times\prod_{p\in S}\mathbb{Q}_p \times \prod_{p\notin S}\mathbb{Z}_p$ space

$$(f * g)(x) = \int_{\mathbb{A}^D}\int_{\mathbb{A}^D} dkdk'\ \chi(-(x^ik_i + x^jk'_j) + \frac{1}{2}k_ik'_j\theta^{ij})\tilde{f}_A(k)\tilde{g}_A(k') \tag{25}$$

Taking into account (5), (22) and the property of the Fourier transform of Ω function, one has

$$\begin{aligned}(f * g)(x) = \chi_\infty&\left(-\frac{\theta^{ij}}{2(2\pi)^2}\frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j}\right)f(y)g(z)|_{y=z=x}\\ \times\prod_{p\in S}\int_{\mathbb{Q}_p^D}\int_{\mathbb{Q}_p^D} dkdk'\ &\chi_p(-(x^ik_i + x^jk'_j) + \frac{1}{2}k_ik'_j\theta^{ij})\tilde{f}_p(k)\tilde{g}_p(k')\\ \times\prod_{p\notin S}&\int_{\mathbb{Z}_p^D}\int_{\mathbb{Z}_p^D} dkdk'\ \chi_p(-(x^ik_i + x^jk'_j) + \frac{1}{2}k_ik'_j\theta^{ij}).\end{aligned} \tag{26}$$

It can be shown that if for all p, $\varphi(x) = \Omega(x)$, the adelic Moyal product becomes real one.

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